2.53. Temperaturen $T = T(r,t)$ hos vatten, som med viss hastighet strömmar ur en punktkälla (origo) och likformigt sprider sig i alla riktningar över xy-planet uppfyller ekvationen (1)

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial r^2} - \frac{1}{r}\frac{\partial T}{\partial r},$$

där t = tid, (x,y) = läge och $r = \sqrt{x^2+y^2}$.

Vi söker lösningar på formen

$$T(r,t) = f\left(\frac{r}{\sqrt{t}}\right)$$

med f funktion av en variabel. Bestäm alla sådana funktioner f.

Lösning:

$$\frac{\partial T}{\partial r} = f'\left(\frac{r}{\sqrt{t}}\right)\cdot\frac{1}{\sqrt{t}}$$

$$\frac{\partial^2 T}{\partial r^2} = f''\left(\frac{r}{\sqrt{t}}\right)\cdot\frac{1}{\sqrt{t}}\cdot\frac{1}{\sqrt{t}} = \frac{1}{t}f''\left(\frac{r}{\sqrt{t}}\right)$$

$$\frac{\partial T}{\partial t} = f'\left(\frac{r}{\sqrt{t}}\right)\cdot r\cdot\left(-\frac{1}{2}\right)t^{-3/2} = \frac{-r}{2t^{3/2}}f'\left(\frac{r}{\sqrt{t}}\right)$$

Vi får nu $\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial r^2} - \frac{1}{r}\frac{\partial T}{\partial r} \iff$

(2)

$$\iff -\frac{r}{2t^{3/2}}f'\left(\frac{r}{\sqrt{t}}\right) = \frac{1}{t}f''\left(\frac{r}{\sqrt{t}}\right) - \frac{1}{r}f'\left(\frac{r}{\sqrt{t}}\right)\cdot\frac{1}{\sqrt{t}}$$

$$\iff f''\left(\frac{r}{\sqrt{t}}\right) - \frac{\sqrt{t}}{r}f'\left(\frac{r}{\sqrt{t}}\right) + \frac{r}{2\sqrt{t}}f'\left(\frac{r}{\sqrt{t}}\right) = 0$$

$$\iff f''\left(\frac{r}{\sqrt{t}}\right) + \left(\frac{r}{2\sqrt{t}} - \frac{\sqrt{t}}{r}\right)f'\left(\frac{r}{\sqrt{t}}\right) = 0$$

Sätt $u = \frac{r}{\sqrt{t}}$

$$\iff f''(u) + \left(\frac{u}{2} - \frac{1}{u}\right)f'(u) = 0$$

Int. faktor: $e^{\frac{u^2}{4} - \ln u}$ $(u>0)$

$$\iff \left(f'(u)\cdot e^{\frac{u^2}{4}-\ln u}\right)' = 0$$

$$\iff f'(u)\cdot e^{\frac{u^2}{4}-\ln u} = C$$

$$\iff f'(u) = Ce^{\ln u - \frac{u^2}{4}} = Ce^{\ln u}\cdot e^{-\frac{u^2}{4}} = Cue^{-\frac{u^2}{4}}$$

$$\iff f(u) = \underbrace{-2C}_{=B}e^{-\frac{u^2}{4}} + D = Be^{-\frac{u^2}{4}} + D$$

Svar: $f(u) = Be^{-\frac{u^2}{4}} + D$, där B och D är godtyckliga konstanter

2.70. Funktionen $f:\mathbb{R}^3 \to \mathbb{R}$ definieras genom (3)

$$f(x,y,z) = (x + xy + yz)e^x.$$

Ange alla stationära punkter till f. Har f någon lokal extrempunkt?

Lösning: Kollar när $\operatorname{grad} f = \bar{0}$:

$$\begin{cases} f'_x = (1+y)e^x + (x+xy+yz)e^x = \\ \quad = (1+x+y+xy+yz)e^x = 0 \\ f'_y = xe^x + ze^x = (x+z)e^x = 0 \\ f'_z = ye^x = 0 \end{cases}$$

$$\iff \begin{cases} 1 + x + y + xy + yz = 0 & (1) \\ x + z = 0 & (2) \\ y = 0 & (3) \end{cases}$$

(3) ger $y=0$ och (2) ger $z=-x$. Insättning i (1) ger då

$$1 + x + 0 + x\cdot 0 + 0\cdot(-x) = 0 \iff x = -1$$

Vi får den enda stationära punkten $(x,y,z) = (-1,0,1)$.

Kollar om $(-1,0,1)$ är lokal extrempunkt: (4)

Taylorutv.

$$f(a+h,b+k,c+l) - f(a,b,c) =$$

$$= \underbrace{f'_x(a,b,c)}_{0}h + \underbrace{f'_y(a,b,c)}_{0}k + \underbrace{f'_z(a,b,c)}_{0}l +$$

$$+ \underbrace{\frac{1}{2}\Big(f''_{xx}(a,b,c)h^2 + f''_{yy}(a,b,c)k^2 + f''_{zz}(a,b,c)l^2 + 2f''_{xy}(a,b,c)hk + 2f''_{xz}(a,b,c)hl + 2f''_{yz}(a,b,c)kl\Big)}_{Q(h,k,l)} + \text{Restterm}$$

Studera $Q(h,k,l)$ i punkten $(a,b,c) = (-1,0,1)$.

$$f''_{xx} = (1+y)e^x + (1+x+y+xy+yz)e^x = (2+x+2y+xy+yz)e^x$$

$$f''_{yy} = 0$$

$$f''_{zz} = 0$$

$$f''_{xy} = e^x + xe^x + ze^x = (1+x+z)e^x$$

$$f''_{xz} = ye^x$$

$$f''_{yz} = e^x$$

Vi får

$$f''_{xx}(-1,0,1) = (2-1+2\cdot 0+(-1)\cdot 0+0\cdot 1)e^{-1} = e^{-1}$$

$$f''_{yy}(-1,0,1) = 0, \quad f''_{zz}(-1,0,1) = 0$$

$$f''_{xy}(-1,0,1) = (1-1+1)e^{-1} = e^{-1}$$

$$f''_{xz}(-1,0,1) = 0, \quad f''_{yz}(-1,0,1) = e^{-1}, \quad \text{så}$$

$$Q(h,k,l) = e^{-1}h^2 + 2e^{-1}hk + 2e^{-1}kl =$$

$$= \underset{\vee\, 0}{e^{-1}}(h^2+2hk+2kl)$$

och $h^2+2hk+2kl = (h+k)^2 - k^2 + 2kl =$

$$= (h+k)^2 - (k^2-2kl) = (h+k)^2 - ((k-l)^2 - l^2) =$$

$$= (h+k)^2 - (k-l)^2 + l^2,$$

olika tecken

så $Q(h,k,l)$ <u>indefinit</u>, ej lokal extrempunkt

2.85. a) Transformera differentialekvationen

$$x^2\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x\partial y} + y^2\frac{\partial^2 f}{\partial y^2} = xy \qquad (*)$$

genom att införa de nya variablerna

$$\begin{cases} u = x \\ v = \frac{x}{y} \end{cases}.$$

b) Lös differentialekvationen!

<u>Lösning (a+b)</u>: $\boxed{f(u,v) = f(u(x,y), v(x,y))}$

Kedjeregeln ger

- $f'_x = f'_u\cdot u'_x + f'_v\cdot v'_x = f'_u\cdot 1 + f'_v\cdot\frac{1}{y}$
  $= f'_u + f'_v\cdot\frac{1}{y}$
- $f'_y = f'_u\cdot u'_y + f'_v\cdot v'_y = f'_u\cdot 0 + f'_v\cdot\left(-\frac{x}{y^2}\right) =$
  $= -\frac{x}{y^2}f'_v$

Sätt $f'_u = g$, $f'_v = h$. Vi har då $\begin{cases} f'_x = g + \frac{1}{y}h \\ f'_y = -\frac{x}{y^2}h \end{cases}$.

- $f''_{xx} = g'_x + \frac{1}{y}h'_x = g'_u\cdot u'_x + g'_v\cdot v'_x +$
  $+\frac{1}{y}(h'_u\cdot u'_x + h'_v\cdot v'_x) = g'_u\cdot 1 + g'_v\cdot\frac{1}{y} +$
  $+\frac{1}{y}\left(h'_u\cdot 1 + h'_v\cdot\frac{1}{y}\right) =$

$$= g'_u + \frac{1}{y}g'_v + \frac{1}{y}h'_u + \frac{1}{y^2}h'_v = f''_{uu} + \frac{1}{y}f''_{uv} + \frac{1}{y}f''_{vu} + \frac{1}{y^2}f''_{vv} =$$

$$= f''_{uu} + \frac{2}{y}f''_{uv} + \frac{1}{y^2}f''_{vv}$$

- $f''_{xy} = g'_y + \left(-\frac{1}{y^2}\right)h + \frac{1}{y}h'_y = g'_u\cdot u'_y + g'_v\cdot v'_y +$
  $+\left(-\frac{1}{y^2}\right)h + \frac{1}{y}(h'_u\cdot u'_y + h'_v\cdot v'_y) =$
  $= g'_u\cdot 0 + g'_v\cdot\left(-\frac{x}{y^2}\right) - \frac{1}{y^2}h + \frac{1}{y}\left(h'_u\cdot 0 + h'_v\cdot\left(-\frac{x}{y^2}\right)\right) =$
  $= -\frac{x}{y^2}g'_v - \frac{1}{y^2}h - \frac{x}{y^3}h'_v = -\frac{x}{y^2}f''_{uv} - \frac{1}{y^2}f'_v - \frac{x}{y^3}f''_{vv}$
- $f''_{yy} = \frac{2x}{y^3}h - \frac{x}{y^2}h'_y = \frac{2x}{y^3}h - \frac{x}{y^2}(h'_u\cdot u'_y + h'_v\cdot v'_y) =$
  $= \frac{2x}{y^3}h - \frac{x}{y^2}\left(h'_u\cdot 0 + h'_v\cdot\left(-\frac{x}{y^2}\right)\right) =$
  $= \frac{2x}{y^3}h + \frac{x^2}{y^4}h'_v = \frac{2x}{y^3}f'_v + \frac{x^2}{y^4}f''_{vv}$

$$(*) \iff x^2\left(f''_{uu} + \frac{2}{y}\cancel{f''_{uv}} + \frac{1}{y^2}\cancel{f''_{vv}}\right) + 2xy\left(-\frac{x}{y^2}\cancel{f''_{uv}} - \frac{1}{y^2}\cancel{f'_v} - \frac{x}{y^3}\cancel{f''_{vv}}\right) + y^2\left(\frac{2x}{y^3}\cancel{f'_v} + \frac{x^2}{y^4}\cancel{f''_{vv}}\right) = xy$$

$$\iff x^2 f''_{uu} = xy \overset{(x\neq 0)}{\iff} f''_{uu} = \frac{y}{x}$$

$$\iff \boxed{f''_{uu} = \frac{1}{v}} \qquad \text{(yes!)}$$

$$f''_{uu} = \frac{1}{v} \iff f'_u = \frac{1}{v}\cdot u + \varphi_1(v)$$

$$\iff f(u,v) = \frac{1}{2v}u^2 + \varphi_1(v)u + \varphi_2(v)$$

$$\iff f(x,y) = \frac{x^2}{2\cdot x/y} + \varphi_1\left(\frac{x}{y}\right)x + \varphi_2\left(\frac{x}{y}\right) =$$

$$= \underline{\underline{\frac{xy}{2} + \varphi_1\left(\frac{x}{y}\right)x + \varphi_2\left(\frac{x}{y}\right)}},$$

där φ_1, φ_2 är godtyckliga C^2-funktioner av en variabel.